Mathematical Properties of the Elasticity Difference Tensor

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Abstract. A tetrad, adapted to the principal directions of the unstrained reference tensor, is chosen and the elasticity difference tensor, as introduced in [1], is decomposed along those directions. The second order tensors obtained are studied and an example is presented.

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INTRODUCTION

Here we will consider a continuous medium possessing elastic properties, the collection of all its idealized particles being the 3-dimensional space X - the material space. (M,g) represents the space-time manifold, i.e. M is a four-dimensional, connected, Hausdorff manifold and g is a Lorentzian metric with signature $(-+++)$ such that $g=-\mathbf{u}\otimes\mathbf{u}+h$, where: (i) $h=\pi^{*}k$, π^{*} being the usual canonical projection onto X and k being a metric in X; (ii) $\pi^{-1}(p\in X)$ defines a timelike curve in M having $\mathbf{u}$ as unit tangent vector field and represents the flowline of p; (iii) $\pi:U\subset M\longrightarrow X$ describes a state of matter.

Following [1], for an unrelaxed state of matter the unstrained reference tensor [2] can be written as $k_{ab}=n_1^2x_ax_b+n_2^2y_ay_b+n_3^2z_az_b$, the scalar fields n_1,n_2,n_3 being related to the eigenvalues along the principal directions of k^a_b. An orthonormal tetrad $\{\mathbf{u,x,y,z}\}$, with $\mathbf{u}$ timelike and $\mathbf{x,y,z}$ unit spacelike vector fields aligned with the eigenvectors of k_a^b, will be used. On a local coordinate system, the metric g can be written as

$$g_{ab}=-u_au_b+x_ax_b+y_ay_b+z_az_b. \tag{1}$$

In order to study elasticity properties of the space-time, the authors in [1] define the elasticity difference tensor:

$$S^a_{bc}=\frac{1}{2}k^{am}(D_bk_{mc}+D_ck_{mb}-D_mk_{bc}),$$

where D denotes projected covariant derivative associated to g. A classification of S will certainly be interesting for the characterization of the elasticity properties of the space-time. In order to do so, we decompose S^a_{bc} along the principal directions of k^a_b:

$$S^a_{bc}=M_{bc}\,x^a+N_{bc}\,y^a+P_{bc}\,z^a.$$

The second order symmetric tensors M,N,P are now investigated.

MAIN RESULTS. AN EXAMPLE

The following results for M_{bc} were obtained, the proofs being in [4].

Theorem 1 *The general form of M_{bc} is given by*

$$M_{bc} = u^m(x_{m;(b}u_{c)} + u_{(b}x_{c);m}) + x_{(b;c)} - x^m x_{(c}x_{b);m} + \gamma_{011}\, u_{(b}x_{c)} - \gamma_{010}\, u_b u_c$$
$$+\frac{1}{n_1}[2n_{1,(b}x_{c)} + 2n_{1,m}u^m u_{(b}x_{c)} + n_{1,m}x^m x_b x_c]$$
$$+\frac{1}{n_1^2}\{-x^m(z_b z_c n_3 n_{3,m} + y_b y_c n_2 n_{2,m}) + n_2^2[(\gamma_{021} - \gamma_{120})u_{(b}y_{c)} + x^m(y_{m;(b}y_{c)} - y_{(b}y_{c);m})]$$
$$+n_3^2[(\gamma_{031} - \gamma_{130})u_{(b}z_{c)} + x^m(z_{m;(b}z_{c)} - z_{(b}z_{c);m})]\},$$

where γ_{abc} are the rotation coefficients and a comma represents a partial derivative.

Theorem 2 ***x*** *is an eigenvector of M_{bc} iff n_1 remains invariant along the directions of* ***y*** *and* ***z****, i.e. $\Delta_y(\log n_1) = \Delta_z(\log n_1) = 0$, where Δ_y represents the intrinsic derivative along* ***y****. The corresponding eigenvalue is $\lambda = \Delta_x(\log n_1)$.*

Theorem 3 ***y*** *is an eigenvector of M_{bc} iff n_1 remains invariant along the direction of* ***y****, i.e. $\Delta_y(\log n_1) = 0$, and $\frac{1}{2}\gamma_{132}[-(n_3^2/n_1^2) + 1] + \frac{1}{2}\gamma_{123}[1 - (n_2^2/n_1^2)] + \frac{1}{2}\gamma_{231}[(n_3^2/n_1^2) - (n_2^2/n_1^2)] = 0$. The corresponding eigenvalue is $\lambda = -(n_2/n_1^2)\Delta_x n_2 + \gamma_{122}[-(n_2^2/n_1^2) + 1]$.*

Theorem 4 ***z*** *is an eigenvector of M_{bc} iff n_1 remains invariant along the direction of* ***z****, i.e. $\Delta_z(\log n_1) = 0$, and $\frac{1}{2}\gamma_{123}[1 - (n_2^2/n_1^2)] + \frac{1}{2}\gamma_{132}[1 - (n_3^2/n_1^2)] + \frac{1}{2}\gamma_{231}[(n_3^2/n_1^2) - (n_2^2/n_1^2)] = 0$. The corresponding eigenvalue is $\lambda = -(n_3/n_1^2)\Delta_x n_3 - \gamma_{133}[(n_3^2/n_1^2) - 1]$.*

Similar results have been obtained by the authors for N and P [4].

The following example illustrates the results above. We consider a spherically symmetric metric g written in local coordinates t, r, θ, ϕ as $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ (see [3], p.186). If a radial deformation is considered such that $ds^2 = -dt^2 + n^2(r)[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2]$, the only non-zero components of the elasticity difference tensor are $S^r_{rr} = S^\theta_{\theta r} = S^\phi_{\phi r} = \frac{1}{n(r)}\frac{dn(r)}{dr}$ and $S^r_{\phi\phi} = -\frac{r^2\sin^2(\theta)}{n(r)}\frac{dn(r)}{dr} = \sin^2(\theta)S^r_{\theta\theta}$. Then $M_{bc} = \lambda_1(x_b x_c - y_b y_c - z_b z_c)$, $N_{bc} = 2\lambda_2(x_b y_c + x_c y_b)$ and $P_{bc} = 2\lambda_3(x_b z_c + x_c z_b)$, where $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{n(r)}\frac{dn(r)}{dr}$. Therefore, the eigenvalue associated with the eigenvector **u** vanishes identically. The remaining eigenvectors are: (i) {**x,y,z**} for M_{bc}, $\frac{1}{n(r)}\frac{dn(r)}{dr}$, $-\frac{1}{n(r)}\frac{dn(r)}{dr}$, $-\frac{1}{n(r)}\frac{dn(r)}{dr}$ being the corresponding eigenvalues, so that the Segre type is $\{1,1(11)\}$; (ii) {**x+y,x-y,z**} for N_{bc} with eigenvalues $\frac{1}{n(r)}\frac{dn(r)}{dr}$, $-\frac{1}{n(r)}\frac{dn(r)}{dr}$ and zero, respectively, the Segre type being $\{1,111\}$; (iii) {**x+z,x-z,y**} for P_{bc} with eigenvalues $\frac{1}{n(r)}\frac{dn(r)}{dr}$, $-\frac{1}{n(r)}\frac{dn(r)}{dr}$ and zero, respectively, the Segre type being then $\{1,111\}$.

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